Function study

1. Why study a function?

The primary goal of a function study is to plot its curve. Even though modern tools (calculator, computer) easily allow this plotting, it is not certain that the displayed result corresponds to reality. Let me explain: a computer tool will always plot a curve by calculating a large number of points, but there is no guarantee that it will not miss a variation finer than its smallest calculation step.

2. The different stages of a function study

A function study is broken down into a number of steps that always follow the same order:

- if the function is periodic, determining the period and choosing the study interval based on this period;
- finding the domain;
- searching for parity which, if it exists, allows the study interval to be reduced to only part of the domain;
- finding the limits at the boundaries of the domain;
- writing the equations of the asymptotes (vertical, horizontal, and oblique);
- calculating the derivative;
- establishing the variation table;
- searching for inflection points;
- establishing a table of values where the coordinates of some notable points are recorded to help plot the curve;
- and thus plotting the curve, which is greatly facilitated by the previously gathered information.

3. Function period

Although one can imagine and create an infinite number of periodic functions, those involved in high school function studies are trigonometric:

- sine and cosine, which have a period of 2π;
- tangent, with a period of π.

If these functions are in the 'simple' form $\sin(x)$ or $\cos(x)$ (for example), one can immediately deduce that the study interval is $[0; 2\pi]$ or $[-\pi; \pi]$ (the important thing is that it spans 2π).

Often, the argument of trigonometric functions is more complex. Consider the function f defined by $f(x) = \cos(3x + 2)$. Here, I need to write:

$$\begin{array}{rrcccl} & -\pi & \leq & 3x+2 & \leq & \pi \\ \Leftrightarrow & -\pi-2 & \leq & x & \leq & \pi-2 \\ \Leftrightarrow & -\frac{\pi+2}{3} & \leq & x & \leq & \frac{\pi-2}{3} \end{array}$$

Thus, the study interval becomes $[-\frac{\pi+2}{3}; \frac{\pi-2}{3}]$.

4. Domain

Finding the domain involves identifying the interval where the function is defined, meaning where there are no forbidden values. Some functions indeed have forbidden values, which are values for which an image cannot be calculated. For example, the square root function cannot be calculated for all strictly negative numbers. To confirm this, one can ask a calculator to give the square root of -5: it will respond with an error message.

There are only four simple rules to remember to determine a domain:

- you cannot calculate the square root of a negative number;
- you cannot divide by zero;
- the argument of a logarithm must be strictly positive;
- a negative number cannot be raised to a negative non-integer power; this is because for any y that is not an integer: $x^y = e^{yln(x)}$.

This last rule tells me that I can calculate, for example, $(-5)^{-2}$ (the calculator gives me 0.04, as it is indeed $\frac{1}{25}$), but not $(-5)^{-2,5}$.

The first rule can be generalized by saying that one cannot calculate the nth root of a negative number for even n. Thus, the expression $\sqrt[3]{-125} = -5$ is allowed, but not $\sqrt[4]{-125}$.

Let's see a practical example to illustrate these rules: let the function f be defined by $f(x) = \ln(x-3) - \frac{1}{\sqrt{x-4}}$.

Given that the argument of a logarithm cannot be negative or zero, we must have:

$$\begin{array}{rccl} & x-3 & > & 0 \\ \Leftrightarrow & x & > & 3 \end{array}$$

This gives me a first interval $]3; +\infty[$. Next, the expression under the square root must be greater than or equal to zero:

$$\begin{array}{rccl} & x-4 & \geq & 0 \\ \Leftrightarrow & x & \geq & 4 \end{array}$$

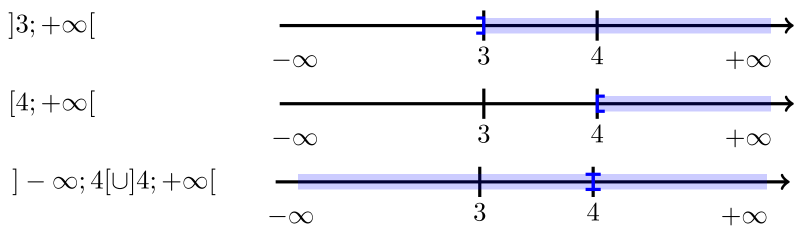
This gives me the second interval: $[4; +\infty[$. I also need to consider the fraction that appears in the expression of f. Its denominator must never be zero, so I write:

$$\begin{array}{rccl} & \sqrt{x-4} & = & 0 \\ \Leftrightarrow & x-4 & = & 0 \\ \Leftrightarrow & x & = & 4 \end{array}$$

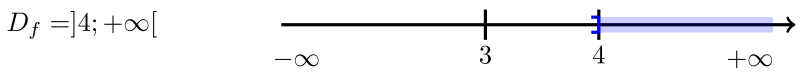
Please note that in moving from the first to the second line, I can keep the equivalence sign by squaring both sides, as they are both positive.

I see that I must exclude 4 from the domain, as it nullifies the denominator. This amounts to considering the interval $]-\infty; 4[\cup]4; +\infty[$.

I now have three intervals at my disposal. Since I can only calculate a value of my function for numbers belonging to each of these three intervals (for example, 4 belongs to the first two but not the last, so it must be excluded), I will take the intersection of these intervals. This will lead me to the domain. To do this, I can represent each interval graphically:



Note the meaning of the brackets: for the first graph, 3 is excluded from the interval, and for the second, 4 is included. Finally, I find the domain D_f by taking the intersection of the previous intervals:



Before concluding this section, just a word on a very particular case to consider: the number 0^0. Mathematicians have debated this number for centuries and still have not reached an agreement! For some, it equals 1, and for others, 0. Therefore, it is wise to exclude this number from a domain if it ever appears (as is the case, for example, with the function defined by $f(x) = x^x$).

5. Parity

Some functions can be even, others odd, and others neither even nor odd, the latter being the majority of functions encountered in high school!

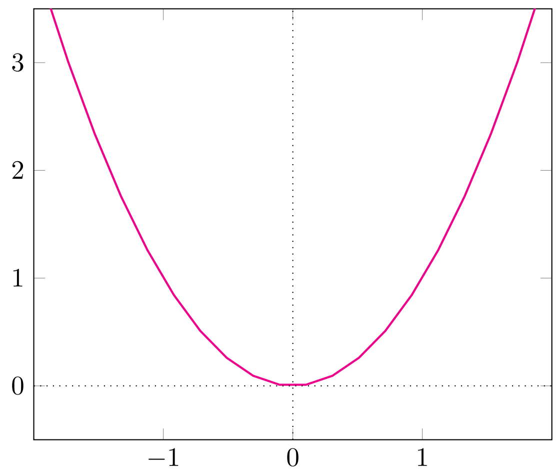
An even function is one such that for all x in its domain: $f(-x) = f(x)$. It follows that the domain must be symmetric with respect to the origin.

An odd function is one such that: $f(-x) = -f(x)$, with the domain also needing to be symmetric with respect to the origin.

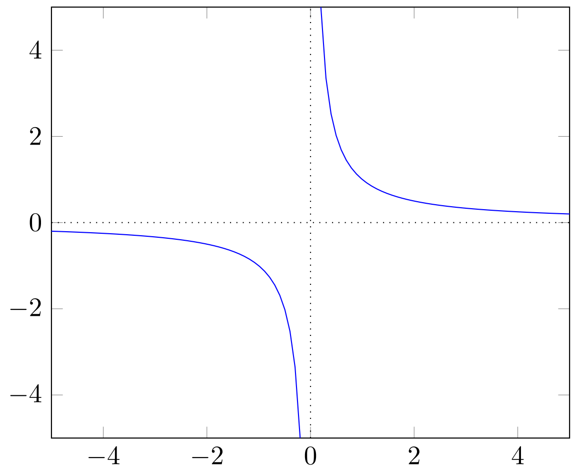
An even function has its curve symmetric with respect to the y-axis, while for an odd function, the curve is symmetric with respect to the origin. Thus, since the goal of studying a function is to plot its curve, I can halve the study interval and deduce the rest of the curve by symmetry. Let's take two examples.

The function f defined by $f(x) = x^2$ is clearly even. Its domain being $\mathbb{R}$, I can reduce the study interval to $[0; +\infty[$.

The function f defined by $f(x) = \frac{1}{x}$ is odd. Its domain is $]-\infty; 0[\cup]0; +\infty[$ (or $\mathbb{R}^*$). I can therefore limit its study to the interval $]0; +\infty[$.



$f(x) = x^2$



$f(x) = \frac{1}{x}$

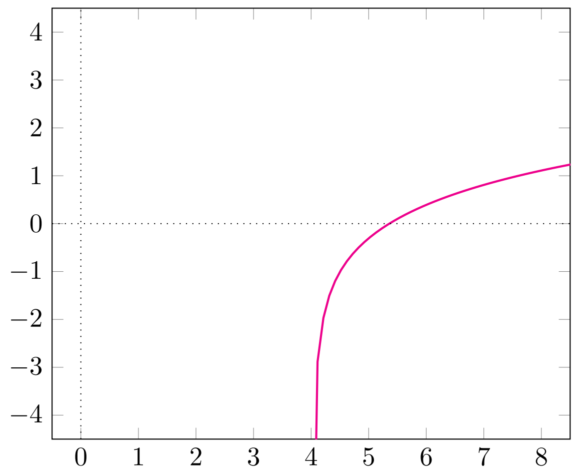
6. Limits at the boundaries of the domain

Functions often exhibit ‘special’ behavior as they approach the boundaries of their domain. For this reason, it is interesting to find the corresponding limits. Additionally, these will be useful when filling out the variation table. Let’s revisit the case of the function f previously defined by $f(x) = \ln(x-3) - \frac{1}{\sqrt{x-4}}$. Knowing that its domain is $D_f =]4; +\infty[$, we seek and find the following limits:

$$\lim_{x \to 4^+} f(x) = -\infty$$
$$\lim_{x \to +\infty} f(x) = +\infty$$

You will note that I calculated the limit from the right of 4 (also called ‘from above’): this is indicated by the presence of the symbol $+$. The reason is that the function is not defined for values less than or equal to 4.

From these two limits, we deduce that first, the curve of f will ‘drop sharply’ as it approaches 4, and secondly, it will ‘rise’ as x becomes larger and larger. This is confirmed by the graph of the function:



$f(x) = \ln(x-3) - \frac{1}{\sqrt{x-4}}$

7. Equations of asymptotes

Asymptotes provide additional information that will make it easier to plot the curve. Therefore, it is useful to obtain their equations.

If they exist, the equations of horizontal and vertical asymptotes are obtained through the limits at the boundaries of the domain. Indeed:

If there exists a such that:

$$\lim_{x\to+\infty} f(x) = a$$

or

$$\lim_{x\to-\infty} f(x) = a$$

then the curve of f has a horizontal asymptote with the equation $y = a$.

Similarly, if there exists b such that:

$$\lim_{x\to b} f(x) = +\infty$$

or

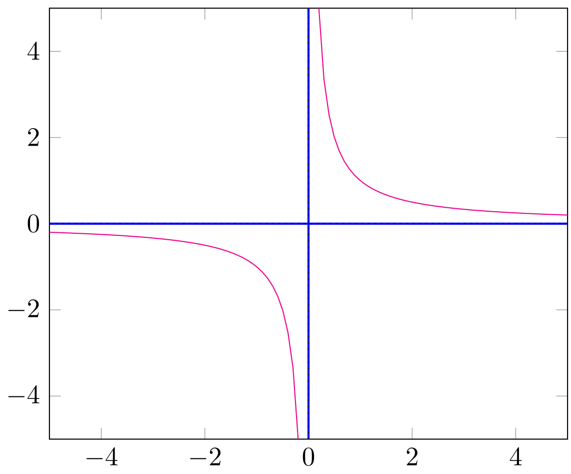
$$\lim_{x\to b} f(x) = -\infty$$

then the curve of f has a vertical asymptote with the equation $x = b$.

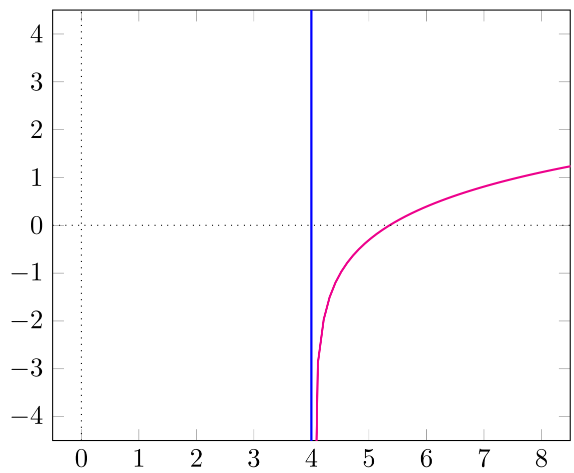
The function f defined by $f(x) = \frac{1}{x}$ and previously discussed has a horizontal asymptote with the equation $y = 0$. Indeed, $\lim\limits_{x\to+\infty} f(x) = 0$ (as well as $\lim\limits_{x\to-\infty} f(x) = 0$).

It also has a vertical asymptote with the equation $x = 0$, because $\lim\limits_{x\to0^+} f(x) = +\infty$ (and also $\lim\limits_{x\to0^-} f(x) = -\infty$).

As for our function f defined by $f(x) = ln(x-3) - \frac{1}{\sqrt{x-4}}$, it has a vertical asymptote with the equation $x = 4$, because $\lim_{x \to 4^+} f(x) = -\infty$ (which I had described qualitatively but not rigorously as 'the curve of f will drop sharply as it approaches 4').



$f(x) = \frac{1}{x}$ and its asymptotes



$f(x) = ln(x-3) - \frac{1}{\sqrt{x-4}}$ and its asymptote

The equations of oblique asymptotes (if they exist) are not immediately deduced from the limits at the boundaries: they require some additional calculations. The first condition for at least one oblique asymptote to exist is that at least one of the following propositions is true:

$$\lim_{x \to +\infty} f(x) = +\infty$$
$$\lim_{x \to +\infty} f(x) = -\infty$$
$$\lim_{x \to -\infty} f(x) = +\infty$$
$$\lim_{x \to -\infty} f(x) = -\infty$$

In the following, I will assume that $\lim_{x \to +\infty} f(x) = +\infty$. It is then necessary to calculate $\lim_{x \to +\infty} \frac{f(x)}{x}$. If this limit exists and is finite, we denote it by a. It is then necessary to calculate $\lim_{x \to +\infty} f(x) - ax$. If this second limit also exists and is finite, we denote it by b. Then the curve of f has the oblique asymptote at $+\infty$ given by the equation:

y=ax+b

The same reasoning can be applied to search for the existence of an oblique asymptote at $-\infty$.

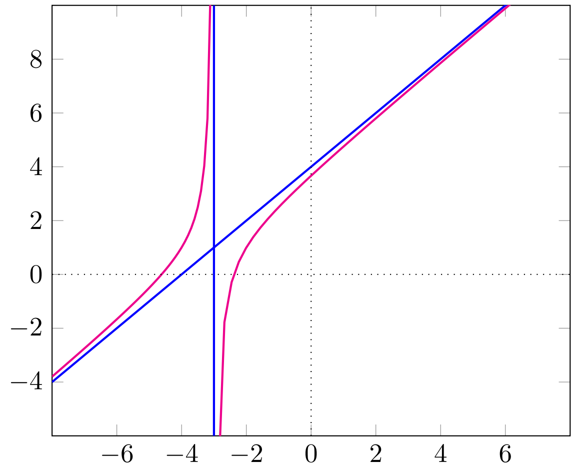
As an example of this latter case, consider the function f defined by $f(x) = \frac{x^2+7x+11}{x+3}$. We find successively:

$$\lim_{x \to +\infty} f(x) = +\infty$$
$$\lim_{x \to +\infty} \frac{f(x)}{x} = 1$$
$$\lim_{x \to +\infty} f(x) - x = 4$$

Therefore, it has the oblique asymptote given by the equation $y = x + 4$ at $+\infty$. It can also be noted that:

$$\lim_{x\to-\infty} f(x) = -\infty$$
$$\lim_{x\to-\infty} \frac{f(x)}{x} = 1$$
$$\lim_{x\to-\infty} f(x) - x = 4$$

Consequently, the same line is also an asymptote at $-\infty$. This is seen in the graph:



$f(x) = \frac{x^2+7x+11}{x+3}$ and its asymptotes

I have also plotted the line with the equation $x = -3$, as it is a vertical asymptote. Indeed:

$$\lim_{x\to-3^+} f(x) = -\infty$$
$$\lim_{x\to-3^-} f(x) = +\infty$$

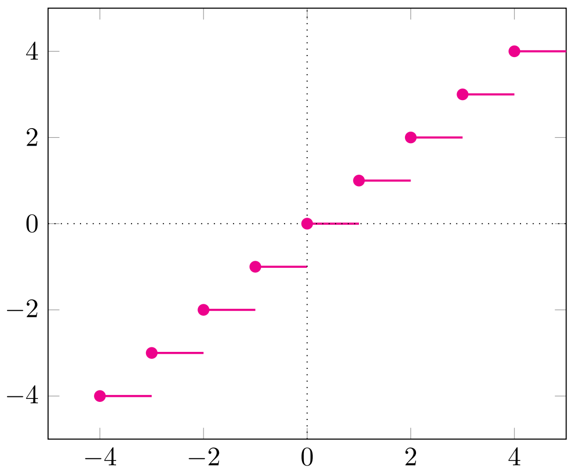
8. Calculating the derivative

The calculation of the derivative is the prelude to the establishment of the variation table. As you certainly know, there are formulas for calculating derivatives. Before addressing them, I will revisit two definitions. The first is the definition of continuity. Indeed, for a function to be differentiable at x_0, it must be continuous at x_0 (this is a necessary but not sufficient condition). We say that f is continuous at x_0 if:

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Thus, the floor function, $f(x) = \lfloor x \rfloor$ (formerly noted $f(x) = E(x)$) is not differentiable at 1, because:

$$\lim_{x\to 1^+} \lfloor x \rfloor = 1$$
$$\lim_{x\to 1^-} \lfloor x \rfloor = 0$$



$f(x) = \lfloor x \rfloor$ (the points belong to the curve)

If the function is continuous at x_0, then it is differentiable at x_0 if the following limit exists and is finite:

$$\lim_{x \to x_0} \frac{f(x) - x_0}{x - x_0}$$

We then denote this limit as $f'(x_0)$. Mathematicians have established (and proven!) formulas to help us easily calculate all kinds of derivatives. The following table summarizes them:

Function	Derivative	Function	Derivative
x	1	ku	ku'
x^n	nx^{n-1}	$u+v$	$u'+v'$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\sqrt{u}$	$\frac{u'}{2\sqrt{u}}$
$\frac{1}{x}$	$\frac{-1}{x^2}$	uv	$u'v+uv'$
$\frac{1}{x^n}$	$\frac{-n}{x^{n+1}}$	$\frac{1}{v}$	$\frac{-v'}{v^2}$
e^x	e^x	$\frac{u}{v}$	$\frac{u'v-uv'}{v^2}$
$\ln(x)$	$\frac{1}{x}$	u^n	$nu'u^{n-1}$
$\cos(ax+b)$	$-\arcsin(ax+b)$	$\frac{1}{u^n}$	$\frac{-nu'}{u^{n+1}}$
$\sin(ax+b)$	$\arccos(ax+b)$	e^u	$u'e^u$
$\tan(ax+b)$	$a[1+\tan^2(ax+b)] = \frac{a}{\cos^2(ax+b)}$	$\ln\lvert u\rvert$	$\frac{u'}{u}$
		$u \circ v(x) = u(v(x))$	$v'(x) \cdot u'(v(x))$

As an example, the function f defined by $f(x) = ln(x-3) - \frac{1}{\sqrt{x-4}}$ has the derivative $f'(x) = \frac{1}{x-3} + \frac{1}{2(\sqrt{x-4})^3} = \frac{1}{x-3} + \frac{1}{2(x-4)^{\frac{3}{2}}}$.

9. Variation table

The variation table is constructed very simply knowing that:

- if on an interval I, $f'(x) \geq 0$, then f is increasing on I;
- if on an interval I, $f'(x) \leq 0$, then f is decreasing on I.

For example, if we consider again the function f defined by $f(x) = \frac{x^2+7x+11}{x+3}$, knowing that its denominator must be non-zero, it is defined on $]-\infty;-3[\cup]-3;+\infty[$, differentiable on this same interval, and its derivative is $f'(x)=\frac{x^2+6x+10}{(x+3)^2}$.

Therefore, to determine the sign of $f'(x)$, since its denominator is always positive, I only need to study the sign of the trinomial $x^2+6x+10$. I start by calculating its discriminant: $\Delta=b^2-4ac=6^2-4\cdot 10=-4$. Since this discriminant is strictly negative, I deduce that $f'(x)$ always has the sign of a, which is positive. This leads to the following variation table:

x	$-\infty$		-3		$+\infty$
$f'(x)$		$+$	$\|\|$	$+$	
$f(x)$	$-\infty$	$\nearrow$	$+\infty$ $\|\|$ $-\infty$	$\nearrow$	$+\infty$

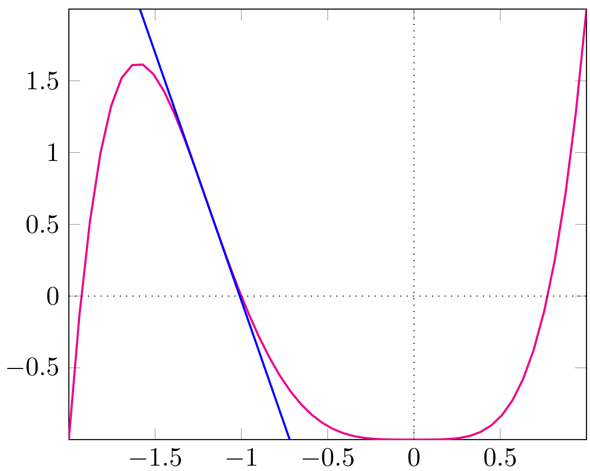
I have transcribed in the table the various limits of f that we have already encountered.

Please note that f is increasing on each of the intervals $]-\infty;-3[$ and $]-3;+\infty[$, but not on the entire $\mathbb{R}$ (it's as if the double bar 'breaks' the interval into two). This can be observed on the previously plotted graph of f.

10. Inflection points

For a function to have an inflection point at a given point, its second derivative must change sign at that point. For example, if we consider the function f defined by $f(x)=x^5+2x^4-1$, its first derivative is $f'(x)=5x^4+8x^3$ and its second derivative is $f''(x)=20x^3+24x^2=4x^2(5x+6)$. The latter changes sign at $x=\frac{-6}{5}$. Note that the second derivative also equals zero at this point, which is logical since it changes sign. It also equals zero at $x=0$, but this does not correspond to an inflection point, as the sign does not change in this second case.

Inflection points result in a 'particular shape' of the curve, as shown in the graph of f:



$f(x)=x^5+2x^4-1$

Indeed, the tangent (in blue) crosses the curve at the inflection point, providing additional information for plotting the latter. The function changes convexity at the inflection point. As a reminder:

- if the tangents on an interval I are all located above the curve of a function f, then f is concave on I;
- if the tangents on an interval I are all located below the curve of a function f, then f is convex on I.

As seen previously, the second derivative also equals zero at 0, but this point is not an inflection point, as the curve is convex before and after it.

Another reminder: the equation of the tangent at a point with coordinates (x_0, y_0) on the curve of f is given by:

$$y = f'(x_0)(x - x_0) + y_0$$

11. Full study

In order to conduct a complete study, we consider the function f defined by $f(x) = \frac{x^2+4x+1}{x+2}$.

Since the denominator must be non-zero, the domain is:

$$Df =]-\infty; -2[\cup]-2; +\infty[\text{ (which can also be written as } \mathbb{R}\backslash\{-2\})$$

Regarding the limits at the boundaries of the domain, we find:

$$\lim_{x\to -2^+} f(x) = -\infty$$
$$\lim_{x\to -2^-} f(x) = +\infty$$
$$\lim_{x\to +\infty} f(x) = +\infty$$
$$\lim_{x\to -\infty} f(x) = -\infty$$

This immediately shows us that the line D_1 with the equation $x = -2$ is a vertical asymptote. We still need to see the existence of oblique asymptotes. We find successively:

$$\lim_{x\to +\infty} \frac{f(x)}{x} = 1$$
$$\lim_{x\to +\infty} f(x) - x = 2$$
$$\lim_{x\to -\infty} \frac{f(x)}{x} = 1$$
$$\lim_{x\to -\infty} f(x) - x = 2$$

Therefore, the line D_2 with the equation $y = x + 2$ is an oblique asymptote at $+\infty$ and $-\infty$.

To establish the variation table, we calculate the derivative:

$$f'(x) = \frac{x^2+4x+7}{(x+2)^2}$$

Since the denominator of $f'(x)$ is always positive, we need to study the sign of the trinomial $x^2 + 4x + 7$. We therefore calculate its discriminant: $\Delta = b^2 - 4ac = 4^2 - 4 \cdot 7 = 16 - 28 = -12$. Given that the latter is always negative, the trinomial has a constant sign and the sign of a. That is to say, it is always positive. We deduce the variation table:

x	$-\infty$	-2	$+\infty$
$f'(x)$	$+$	‖	$+$
$f(x)$	$-\infty \nearrow +\infty$	‖	$-\infty \nearrow +\infty$

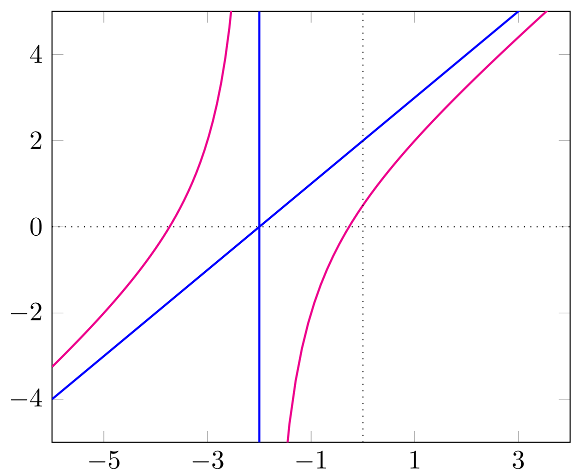
This table shows us that the curve crosses the x-axis twice. It is interesting to know the x-coordinates of these two intersection points for plotting. To find them, we need to solve the equation $f(x) = 0$, which simplifies to $x^2 + 4x + 1 = 0$. The formulas for solving quadratic equations give us successively:

$$\Delta = b^2 - 4ac = 4^2 - 4 = 12 > 0$$
$$x_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{-4-\sqrt{12}}{2} = -\sqrt{3} - 2$$
$$x_2 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{-4+\sqrt{12}}{2} = \sqrt{3} - 2$$

We thus obtain the following table of values:

x	$-\sqrt{3} - 2 \approx -3.73$	$\sqrt{3} - 2 \approx -0.267$
$f(x)$	0	0

We now have all the information to make the tracing easier:



$f(x) = \frac{x^2+4x+1}{x+2}$

12. Conclusion

We have seen that studying a function, with the aim of plotting its graph, can be done easily if we follow the method we have outlined. To help you verify your work, the site lovemaths.eu will provide a broad study of most functions you may encounter in high school.